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Fourth-Order Approximation for the Rotational Distortion of Stars of Arbitrary Structure

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Clairaut's theory of the rotational distortion of self-gravitating configurations of arbitrary structure, arising from axial rotation with constant angular velocity, previously developed to quantities of third order in superficial distortion, has now been extended to fourth-order terms. The differential equations governing the form and exterior potential of stars rotating in the previously described manner have been set up by the method followed in a previous paper [Z. Kopal, *Astrophys. Space Sci.* 24, 145 (1973)] together with their boundary conditions. Their applications to practical cases are being postponed for a subsequent investigation.

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FOURTH-ORDER APPROXIMATION FOR THE ROTATIONAL DISTORTION OF STARS OF ARBITRARY STRUCTURE

Zdeněk Kopal and M. Kamala Mahanta*

INTRODUCTION

In a previous investigation [1], hereafter referred to as Paper 1, a new method was developed for establishing the shape and gravitational potential of an equilibrium configuration of arbitrary structure, rotating with constant angular velocity,† whose free surface constitutes an equipotential. This method, whose first approximation is well known under the name of Clairaut's theory, is susceptible of extension to any desired order of accuracy; in Paper 1 it was explicitly extended to the quantities of third order in superficial distortion of the rotating configuration.

An application of this theory to the evaluation of the potential energy of the respective configuration required to ascertain the limits of its secular stability disclosed, however, that for homogeneous configurations the errors inherent in a third-order approximation are still too large to specify the limits of stability with satisfactory precision. Although for centrally condensed configurations this precision is likely to be much higher, an extension of the accuracy of our procedure to terms of fourth order appears to be desirable. The aim of the present investigation will be to accomplish this task and to generalize the results of Paper 1 consistently to quantities of fourth order.

To do so in a minimum of space, the principal features of the method employed will no longer be repeated herein; the reader desirous to get acquainted with them should consult Paper 1. Moreover, all notations employed in the present investigation will be made strictly consistent with those of Paper 1, so that no explanations need generally be given. What we shall do will be merely to augment the requisite equations by fourth-order terms, so that a development of the final results to be given in this report will be internally consistent; their numerical applications to problems arising in astrophysics and planetology will be left to future publications.

EQUATIONS OF THE PROBLEM

By a method developed in section III of Paper 1 (and by an obvious extension of the expansions represented by Eqs. 4.2–4.4 of that paper), it is possible to show that if the symbolic expression for the radius vector r' is written as

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†It should be stressed again that this method remains equally applicable to variable angular velocity $\omega(r, \theta)$ as well, provided only that the latter can be expressed in the form of a series factored by zonal harmonics $P_j(\cos\theta)$ of ascending orders.

Note: Manuscript submitted July 18, 1974.

$$r' = a\{1 + f_0 + f_2P_2(\cos\theta) + f_4P_4(\cos\theta) + f_6P_6(\cos\theta) + f_8P_8(\cos\theta) + \ldots\}\,, \quad (2.1)$$

it is possible to show that, correctly to quantities of fourth order, the coefficients E_j and $F_j (j = 0, 2, 4, 6, 8)$ introduced by Eqs. (3.35) and (3.36) of Paper 1 assume the forms

$$E_0 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{a^2\left[\frac{1}{2} - \frac{1}{10} f_2^2 - \frac{2}{105} f_2^3 + \frac{1}{50} f_2^4 - \frac{1}{18} f_4^2 - \frac{2}{35} f_2^2 f_4\right]\right\} da\,, \quad (2.2)$$

$$E_2 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{f_2 - \frac{1}{7} f_2^2 + \frac{12}{35} f_2^3 - \frac{119}{1155} f_2^4 - \frac{50}{693} f_4^2 - \frac{2}{7} f_2 f_4 + \frac{12}{77} f_2^2 f_4\right\} da\,, \quad (2.3)$$

$$E_4 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{\frac{1}{a^2}\left[f_4 - \frac{27}{35} f_2^2 + \frac{216}{385} f_2^3 - \frac{38394}{25025} f_2^4 - \frac{243}{1001} f_4^2 - \frac{60}{77} f_2 f_4 - \frac{35}{143} f_2 f_6 + \frac{13737}{5005} f_2^2 f_4\right]\right\} da\,, \quad (2.4)$$

$$E_6 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{\frac{1}{a^4}\left[f_6 + \frac{90}{77} f_2^3 - \frac{18}{11} f_2^4 - \frac{50}{99} f_4^2 - \frac{25}{11} f_2 f_4 - \frac{14}{11} f_2 f_6 + \frac{270}{77} f_2^2 f_4\right]\right\} da\,, \quad (2.5)$$

$$E_8 = \int_a^{a_1} \rho \frac{\partial}{\partial a}\left\{\frac{1}{a^6}\left[f_8 - \frac{1512}{715} f_2^4 - \frac{1715}{1287} f_4^2 - \frac{196}{65} f_2 f_6 + \frac{784}{143} f_2^2 f_4\right]\right\} da \quad (2.6)$$

and

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$$F_0 = \int_0^a \rho a^2 \, da \,, \tag{2.7}$$

$$F_2 = \int_0^a \rho \frac{\partial}{\partial a} \left\{ a^5 \left[f_2 + \frac{4}{7} f_2^2 + \frac{2}{35} f_2^3 - \frac{184}{1155} f_2^4 + \frac{200}{693} f_4^2 + \frac{8}{7} f_2 f_4 + \frac{72}{77} f_2^2 f_4 \right] \right\} da \,, \tag{2.8}$$

$$F_4 = \int_0^a \frac{\partial}{\partial a} \left\{ a^7 \left[f_4 + \frac{54}{35} f_2^2 + \frac{108}{77} f_2^3 + \frac{1458}{5005} f_2^4 + \frac{486}{1001} f_4^2 + \frac{120}{77} f_2 f_4 + \frac{270}{143} f_2 f_6 + \frac{20829}{5005} f_2^2 f_4 \right] \right\} da \,, \tag{2.9}$$

$$F_6 = \int_0^a \rho \frac{\partial}{\partial a} \left\{ a^9 \left[f_6 + \frac{24}{11} f_2^3 + \frac{144}{55} f_2^4 + \frac{80}{99} f_4^2 + \frac{40}{11} f_2 f_4 + \frac{112}{55} f_2 f_6 + \frac{72}{11} f_2^2 f_4 \right] \right\} da \,, \tag{2.10}$$

$$F_8 = \int_0^a \rho \frac{\partial}{\partial a} \left\{ a^{11} \left[f_8 + \frac{432}{143} f_2^4 + \frac{2450}{1287} f_4^2 + \frac{56}{13} f_2 f_6 + \frac{1260}{143} f_2^2 f_4 \right] \right\} da \tag{2.11}$$

as a fourth-order generalization of Eqs. (4.13)-(4.20) of Paper 1, with

$$f_0 = -\frac{1}{5} f_2^2 - \frac{2}{105} f_2^3 - \frac{1}{9} f_4^2 - \frac{2}{35} f_2^2 f_4 \tag{2.12}$$

in order to render the mass of the rotating configuration (F_0) constant correctly to quantities of fourth order.

As the next step of our analysis, we expand again the total potential $\psi(r') = U + V + V'$ in a Neumann series in terms of the spherical harmonics $P_j(\cos\theta)$ and equate their coefficients α_j to zero for j = 2, 4, 6, and 8. In particular, since the disturbing potential $V'(r') = (1/2)\omega^2 r'^2 \sin^2\theta$ due to centrifugal force can be expanded in a series of the form

$$
\begin{aligned}
V'(r') = {} & \frac{1}{3}\,\omega^2 a^2 \left\{1 - \frac{2}{5} f_2 - \frac{9}{35} f_2^2 + \frac{22}{525} f_2^3 - \frac{4}{35} f_2 f_4\right\} \\
& - \frac{1}{3}\,\omega^2 a^2 \left\{1 - \frac{10}{7} f_2 - \frac{9}{35} f_2^2 + \frac{26}{105} f_2^3 \right. \\
& \left. + \frac{4}{7} f_4 - \frac{20}{77} f_2 f_4 \right\} P_2(\cos\theta) \\
& - \frac{2}{3}\,\omega^2 a^2 \left\{\frac{18}{35} f_2 - \frac{9}{77} f_2^2 - \frac{18}{175} f_2^3 - \frac{57}{77} f_4 \right. \\
& \left. + \frac{489}{5005} f_2 f_4 + \frac{45}{143} f_6 \right\} P_4(\cos\theta) \\
& - \frac{2}{3}\,\omega^2 a^2 \left\{\frac{9}{77} f_2^2 + \frac{5}{11} f_4 - \frac{17}{77} f_2 f_4 - \frac{41}{55} f_6\right\} P_6(\cos\theta) \\
& - \frac{2}{3}\,\omega^2 a^2 \left\{\frac{28}{143} f_2 f_4 + \frac{28}{65} f_6\right\} P_8(\cos\theta) + \ldots .
\end{aligned}
\tag{2.13}
$$

Eqs. (4.21)-(4.23) of Paper 1 can be generalized readily with the result that, correctly to quantities of fourth order,

$$
\begin{aligned}
& \frac{a^2 E_2}{5} \left\{1 + \frac{4}{7} f_2 + \frac{1}{35} f_2^2 - \frac{16}{105} f_2^3 + \frac{4}{7} f_4 + \frac{24}{77} f_2 f_4\right\} \\
& + \frac{a^4 E_4}{9} \left\{\frac{8}{9} f_2 + \frac{72}{77} f_2^2 + \frac{400}{693} f_4\right\} \\
& - \frac{F_0}{a} \left\{f_2 - \frac{2}{7} f_2^2 + \frac{29}{35} f_2^3 \right. \\
& \left. - \frac{454}{1155} f_2^4 - \frac{100}{693} f_4^2 - \frac{4}{7} f_2 f_4 + \frac{36}{77} f_2^2 f_4 \right\} \\
& + \frac{F_2}{5a^3} \left\{1 - \frac{6}{7} f_2 + \frac{111}{35} f_2^2 - \frac{1242}{385} f_2^3 - \frac{6}{7} f_4 + \frac{144}{77} f_2 f_4\right\} \\
& - \frac{F_4}{9a^5} \left\{\frac{10}{7} f_2 + \frac{500}{693} f_4 - \frac{180}{77} f_2^2\right\} \\
& = \frac{\omega^2 a^2}{12\pi G} \left\{1 + \frac{10}{7} f_2 - \frac{9}{35} f_2^2 + \frac{26}{105} f_2^3 + \frac{4}{7} f_4 - \frac{20}{77} f_2 f_4\right\},
\end{aligned}
\tag{2.14}
$$

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$$\frac{a^2E_2}{5}\left\{\frac{36}{35}f_2+\frac{108}{385}f_2^2-\frac{36}{175}f_2^3+\frac{40}{77}f_4+\frac{90}{143}f_6+\frac{3578}{5005}f_2f_4\right\}$$

$$+\frac{a^2E_4}{9}\left\{1+\frac{80}{77}f_2+\frac{1346}{1001}f_2^2+\frac{648}{1001}f_4\right\}+\frac{a^6E_6}{13}\left\{\frac{270}{143}f_2\right\}$$

$$+\frac{F_0}{a}\left\{\frac{18}{35}f_2^2-\frac{108}{385}f_2^3+\frac{16902}{25025}f_2^4-f_4\right.$$

$$\left.+\frac{40}{77}f_2f_4+\frac{90}{143}f_2f_6-\frac{7369}{5005}f_2^2f_4+\frac{162}{1001}f_4^2\right\}$$

$$-\frac{F_2}{5a^3}\left\{\frac{54}{35}f_2-\frac{648}{385}f_2^2+\frac{122688}{25025}f_2^3\right.$$

$$\left.-\frac{21468}{5005}f_2f_4+\frac{60}{77}f_4+\frac{135}{143}f_6\right\}$$

$$+\frac{F_4}{9a^5}\left\{1-\frac{100}{77}f_2+\frac{6368}{1001}f_2^2-\frac{810}{1001}f_4\right\}-\frac{F_6}{13a^7}\left\{\frac{315}{143}f_2\right\}$$

$$=\frac{\omega^2a^2}{6\pi G}\left\{\frac{18}{35}f_2-\frac{9}{77}f_2^2-\frac{18}{175}f_2^3\right.$$

$$\left.-\frac{57}{77}f_4-\frac{45}{143}f_6+\frac{489}{5005}f_2f_4\right\}, \tag{2.15}$$

$$\frac{a^2E_2}{5}\left\{\frac{10}{11}f_4+\frac{28}{55}f_6+\frac{18}{77}f_2^2+\frac{36}{77}f_2f_4\right\}$$

$$+\frac{a^4E_4}{9}\left\{\frac{20}{11}f_2+\frac{108}{77}f_2^2+\frac{80}{99}f_4\right\}+\frac{a^6E_6}{13}\left\{1+\frac{84}{55}f_2\right\}$$

$$-\frac{F_0}{a}\left\{\frac{18}{77}f_2^3-\frac{72}{385}f_2^4-\frac{10}{11}f_2f_4\right.$$

$$\left.+\frac{54}{77}f_2^2f_4-\frac{20}{99}f_4^2+f_6-\frac{28}{55}f_2f_6\right\}$$

$$+\frac{F_2}{5a^3}\left\{\frac{108}{77}f_2^2-\frac{144}{77}f_2^3-\frac{15}{11}f_4+\frac{216}{77}f_2f_4-\frac{42}{55}f_6\right\}$$

$$-\frac{F_4}{9a^5}\left\{\frac{25}{11}f_2+\frac{100}{99}f_4-\frac{270}{77}f_2f_4\right\} \tag{2.16}$$

Continued

$$+\frac{F_6}{13a^7}\left\{1-\frac{98}{55}f_2\right\}$$

$$=\frac{\omega^2a^2}{6\pi G}\left\{\frac{9}{77}f_2^2+\frac{5}{11}f_4-\frac{17}{77}f_2f_4-\frac{41}{55}f_6\right\}, \tag{2.16}$$

and

$$\frac{a^2E_2}{5}\left\{\frac{56}{65}f_6+\frac{56}{143}f_2f_4\right\}+\frac{a^4E_4}{9}\left\{\frac{168}{143}f_2^2+\frac{1960}{1287}f_4\right\}$$

$$+\frac{a^6E_6}{13}\left\{\frac{168}{65}f_2\right\}+\frac{a^8E_8}{17}$$

$$+\frac{F_0}{a}\left\{\frac{72}{715}f_2^4-\frac{84}{143}f_2^2f_4+\frac{490}{1287}f_4^2+\frac{56}{65}f_2f_6-f_8\right\}$$

$$-\frac{F_2}{5a^3}\left\{\frac{144}{143}f_2^3-\frac{336}{143}f_2f_4+\frac{84}{65}f_6\right\}$$

$$+\frac{F_4}{9a^5}\left\{\frac{420}{143}f_2^2-\frac{2450}{1287}f_4\right\}-\frac{F_6}{13a^7}\left\{\frac{196}{65}f_2\right\}+\frac{F_8}{17a^9}$$

$$=\frac{\omega^2a^2}{6\pi G}\left\{\frac{28}{65}f_6+\frac{28}{143}f_2f_4\right\}. \tag{2.17}$$

Next, let us eliminate from the foregoing equations the quantities $E_{2j}(j = 2, 4, 6, 8)$ wherever they are multiplied by the f_j's. We may note that, correctly to quantities of first order, E_2 can be solved for from Eq. (4.24) of Paper 1; Eqs. (4.25) and (4.27) lend themselves for the same purpose, correctly to quantities of second and third order. Moreover, Eqs. (4.26) and (4.28) can do the same for E_4 as can lastly Eq. (4.29) for E_6. By appropriate insertion from Eqs. (4.24)-(4.29) of Paper 1 in Eqs. (2.14)-(2.17) of this report, we can rewrite the latter in the following alternative forms:

$$\frac{a^2E_2}{5}+\frac{F_2}{5a^3}-\frac{f_2F_0}{a}$$

$$=\frac{F_0}{a}\left\{-\frac{6}{7}f_2^2+\frac{748}{245}f_2^3-\frac{190028}{56595}f_2^4-\frac{16}{7}f_2f_4\right.$$

$$\left.+\frac{2308}{539}f_2^2f_4-\frac{500}{693}f_4^2\right\}$$

$$+\frac{F_2}{5a^3}\left\{\frac{10}{7}f_2-\frac{338}{49}f_2^2\right. \tag{2.18}$$

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$$+\frac{123440}{11319} f_2^3 + \frac{10}{7} f_4 - \frac{3320}{539} f_2 f_4\Bigg\}$$

$$+\frac{F_4}{9a^5}\left\{\frac{18}{7} f_2 - \frac{2988}{539} f_2^2 + \frac{100}{77} f_4\right\}$$

$$+\frac{\omega^2 a^2}{12\pi G}\left\{1 - 2f_2 + \frac{6}{7} f_2^2 - \frac{88}{49} f_2^3 + \frac{20}{7} f_2 f_4\right\}, \tag{2.18}$$

$$\frac{a^4 E_4}{9} + \frac{F_4}{9a^7} - \frac{f_4 F_0}{a}$$

$$= \frac{F_0}{a}\Bigg\{- \frac{54}{35} f_2^2 + \frac{6696}{2695} f_2^3 - \frac{24745554}{2697695} f_2^4$$

$$- \frac{160}{77} f_2 f_4 + \frac{4714257}{385385} f_2^2 f_4 - \frac{810}{1001} f_4^2 - \frac{450}{143} f_2 f_6\Bigg\}$$

$$+\frac{F_2}{5a^3}\Bigg\{\frac{18}{7} f_2 + \frac{100}{77} f_4 - \frac{2988}{539} f_2^2$$

$$+\frac{225}{143} f_6 - \frac{1009016}{77077} f_2 f_4 + \frac{5131332}{207515} f_2^3\Bigg\}$$

$$+\frac{F_4}{9a^5}\left\{\frac{180}{77} f_2 - \frac{6865524}{385385} f_2^2 + \frac{1458}{1001} f_4\right\}$$

$$+\frac{F_6}{13a^7}\left\{\frac{45}{11} f_2\right\}$$

$$+\frac{\omega^2 a^2}{12\pi G}\left\{\frac{54}{35} f_2^2 - \frac{5184}{2695} f_2^3 - 2f_4 + \frac{200}{77} f_2 f_4\right\}, \tag{2.19}$$

$$\frac{a^6 E_6}{13} + \frac{F_6}{13a^7} - \frac{f_6 F_0}{a}$$

$$= \frac{F_0}{a}\Bigg\{\frac{216}{77} f_2^3 - \frac{39276}{5929} f_2^4 - \frac{40}{11} f_2 f_4$$

$$+\frac{8630}{847} f_2^2 f_4 - \frac{100}{99} f_4^2 - \frac{28}{11} f_2 f_6\Bigg\}$$

$$+\frac{F_2}{5a^3}\Bigg\{\frac{25}{11} f_4 - \frac{9780}{847} f_2 f_4$$

$$-\frac{450}{77} f_2^2 + \frac{100404}{5929} f_2^3 + \frac{14}{11} f_6\Big\}$$

$$+\frac{F_4}{9a^5}\left\{\frac{45}{11} f_2 - \frac{10674}{847} f_2^2 + \frac{20}{11} f_4\right\}$$

$$+\frac{F_6}{13a^7}\left\{\frac{182}{55} f_2\right\}$$

$$+\frac{\omega^2 a^2}{12\pi G}\left\{\frac{50}{11} f_2 f_4 - \frac{180}{77} f_2^3 - 2f_6\right\}, \tag{2.20}$$

and

$$\frac{a^8 E_8}{17} + \frac{F_8}{17a^9} - \frac{f_8 F_0}{a}$$

$$= \frac{F_0}{a}\left\{-\frac{72}{13} f_2^4 + \frac{140}{13} f_2^2 f_4 - \frac{2450}{1287} f_4^2 - \frac{56}{13} f_2 f_6 + f_8\right\}$$

$$+\frac{F_2}{5a^3}\left\{\frac{144}{11} f_2^3 - \frac{1680}{143} f_2 f_4 + \frac{28}{13} f_6\right\}$$

$$+\frac{F_4}{9a^5}\left\{\frac{490}{143} f_4 - \frac{1764}{143} f_2^2\right\} + \frac{F_6}{13a^7}\left\{\frac{28}{5} f_2\right\}, \tag{2.21}$$

generalizing Eqs. (4.27)-(4.29) of Paper 1 to an accuracy of fourth order.

To proceed further, multiply the foregoing Eqs. (2.18)-(2.21) by a^{-j}, ($j = 2, 4, 6, 8$) so as to render the coefficients of E_j on their left-hand sides constant, and differentiate with respect to a. The derivatives of F_j are merely equal to the integrands on the right-hand sides of Eqs. (2.7)-(2.11); those of E_j are equal to the integrands in Eqs. (2.2)-(2.6) taken with the negative sign (since the independent variable a occurs in the lower limit of the definite integrals for E_j). If subsequently we eliminate the terms F_n for $n \neq j$ (factored by small quantities) with the aid of Eqs. (4.24)-(4.29) of Paper 1, valid to the accuracy of lower orders and similarly treated, it is possible to show that, correctly to the accuracy of fourth order,

$$F_2(a) = a^2\Big\{(3-\eta_2)f_2 + \frac{2}{7}\left(6 - 2\eta_2 + \eta_2^2\right)f_2^2$$

$$-\frac{2}{35}\left(51 + 5\eta_2 + 10\eta_2^2 + 4\eta_2^3\right)f_2^3$$

$$-\frac{8}{1155}\left(231 - 110\eta_2 + 51\eta_2^2 - 66\eta_2^3 - 21\eta_2^4\right)f_2^4 \tag{2.22}$$

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$$
\begin{aligned}
&+\frac{4}{7}\left(13-\eta_2-\eta_4+\eta_2\eta_4\right)f_2f_4 + \frac{4}{77}\left(48-46\eta_2+3\eta_2^2\right.\\
&\left.+6\eta_2\eta_4-9\eta_2^2\eta_4-2\eta_4\right)f_2^2f_4\\
&\left.+\frac{100}{693}\left(20-2\eta_4+\eta_2^2\right)f_4^2\right\} F_0\\
&+\frac{\omega^2a^5}{6\pi G}\left\{\eta_2f_2 + \frac{2}{7}\left(2-\eta_2\right)\eta_2f_2^2\right.\\
&+\frac{2}{35}\left(54+5\eta_2+10\eta_2^2+4\eta_2^3\right)f_2^3\\
&\left.-\frac{4}{7}\left(7-\eta_2-\eta_4+\eta_2\eta_4\right)f_2f_4\right\}, \qquad (2.22)
\end{aligned}
$$

$$
\begin{aligned}
F_4(a) = a^4 &\left\{\left(5-\eta_4\right)f_4 + \frac{18}{35}\left(6-4\eta_2+\eta_2^2\right)f_2^2\right.\\
&+\frac{36}{385}\left(27-17\eta_2+9\eta_2^2-3\eta_2^3\right)f_2^3\\
&-\frac{18}{25025}\left(3804-1052\eta_2+874\eta_2^2-124\eta_2^3-367\eta_2^4\right)f_2^4\\
&+\frac{40}{77}\left(13-2\eta_2-2\eta_4+\eta_2\eta_4\right)f_2f_4\\
&+\frac{1}{5005}\left(40827-28750\eta_2-7179\eta_4\right.\\
&\left.+362\eta_2^2+8732\eta_2\eta_4-4366\eta_2^2\eta_4\right)f_2^2f_4\\
&+\frac{90}{143}\left(24-2\eta_2-2\eta_6+\eta_2\eta_6\right)f_2f_6\\
&\left.+\frac{162}{1001}\left(20-4\eta_4+\eta_4^2\right)f_4^2\right\}F_0\\
&+\frac{\omega^2a^7}{6\pi G}\left\{-\left(2-\eta_4\right)f_4 + \frac{18}{35}\left(3+4\eta_2-\eta_2^2\right)f_2^2\right.\\
&+\frac{36}{385}\left(18+17\eta_2-9\eta_2^2+3\eta_2^3\right)f_2^3\\
&\left.-\frac{40}{77}\left(4-2\eta_2-2\eta_4+\eta_2\eta_4\right)f_2f_4\right\}, \qquad (2.23)
\end{aligned}
$$

$$F_6(a) = a^6 \Bigg\{ \left(7 - \eta_6\right) f_6 + \frac{18}{77} \left(15 - 11\eta_2 + 5\eta_2^2 - \eta_2^3\right) f_2^3$$

$$+ \frac{36}{385} \left(40 + 28\eta_2 + 17\eta_2^2 - 8\eta_2^3 + 2\eta_2^4\right) f_2^4$$

$$+ \frac{10}{11} \left(13 - 3\eta_2 - 3\eta_4 + \eta_2\eta_4\right) f_2 f_4$$

$$+ \frac{2}{77} \left(720 - 226\eta_2 + 45\eta_2^2 - 134\eta_4 + 90\eta_2\eta_4 - 27\eta_2^2\eta_4\right) f_2^2 f_4$$

$$+ \frac{20}{99} \left(20 - 6\eta_4 + \eta_4^2\right) f_4^2 + \frac{28}{55} \left(24 - 3\eta_2 - 3\eta_6 + \eta_2\eta_6\right) f_2 f_6 \Bigg\} F_0$$

$$- \frac{\omega^2 a^9}{6\pi G} \Bigg\{ \left(4 - \eta_6\right) f_6 - \frac{18}{77} \left(13 + 11\eta_2 - 5\eta_2^2 + \eta_2^3\right) f_2^3$$

$$+ \frac{10}{11} \left(1 - 3\eta_2 - 3\eta_4 + \eta_2\eta_4\right) f_2 f_4 \Bigg\} , \tag{2.24}$$

and

$$F_8(a) = a^8 \Bigg\{ \left(9 - \eta_8\right) f_8 + \frac{56}{65} \left(24 - 4\eta_2 - 4\eta_6 + \eta_2\eta_6\right) f_2 f_6$$

$$+ \frac{490}{1287} \left(20 - 8\eta_4 + \eta_4^2\right) f_4^2$$

$$+ \frac{28}{143} \left(112 - 40\eta_2 + 7\eta_2^2 - 24\eta_4 + 14\eta_2\eta_4 - 3\eta_2^2\eta_4\right) f_2^2 f_4$$

$$+ \frac{72}{715} \left(42 - 32\eta_2 + 17\eta_2^2 - 6\eta_2^3 + \eta_2^4\right) f_2^4 \Bigg\} F_0 , \tag{2.25}$$

where F_0 continues to be given by Eq. (2.7) and where (as in Paper 1),

$$\eta_j = \frac{a}{f_j} \frac{\partial f_j}{\partial a} \tag{2.26}$$

denotes the logarithmic derivative of the individual amplitudes f_j on the right-hand side of the expansion Eq. (2.1).

As the last step of preparatory analysis, let us equate Eqs. (2.22)-(2.25) for $F_j(a)$ with Eqs. (2.7)-(2.11) and differentiate both sides of the resulting equations for $j = 2, 4, 6, 8$. The outcome discloses that, correctly to quantities of fourth order, the amplitudes $f_j(a)$ of jth harmonic distortion should satisfy the differential equations

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$$a^2 f_2'' + 6D\left(af_2' + f_2\right) - 6f_2$$

$$= \frac{2}{7}\left\{2\eta_2\left(\eta_2 + 9\right) - 9D\eta_2(\eta_2 + 2)\right\} f_2^2$$

$$- \frac{4}{35}\left\{7\eta_2^3 + 33\eta_2^2 + 180\eta_2 + 66 + 3D\left(2\eta_2^3 - 15\eta_2^2 - 27\eta_2 + 5\right)\right\} f_2^3$$

$$+ \frac{4}{7}\left\{2\left(\eta_2\eta_4 + 15\eta_2 + 8\eta_4\right) - 3D\left(3\eta_2\eta_4 + 3\eta_2 + 3\eta_4 - 7\right)\right\} f_2 f_4$$

$$- \frac{4}{385}\left\{2\left(44\eta_2^3 + 397\eta_2^2 + 27\eta_2 - 258\right)\right.$$

$$\left. -3D\left(44\eta_2^3 + 195\eta_2^2 - 66\eta_2 - 136\right)\right\} f_2^4$$

$$+ \frac{4}{77}\left\{2\left(31\eta_2\eta_4 - 37\eta_4 + 47\eta_2^2 - 137\eta_2 - 110\right)\right.$$

$$\left. -9D\left(3\eta_2^2\eta_4 + 6\eta_2\eta_4 + 3\eta_2^2 - 34\eta_2 + 4\eta_4 - 10\right)\right\} f_2^2 f_4$$

$$+ \frac{100}{693}\left\{2\eta_4\left(\eta_4 + 37\right) - 3D\left(3\eta_4^2 + 6\eta_4 - 14\right)\right\} f_4^2$$

$$+ \frac{3\omega^2}{\pi G\bar{\rho}}(1 - D)\left\{(\eta_2 + 1)f_2 + \frac{3}{7}\eta_2\left(\eta_2 + 2\right)f_2^2\right.$$

$$+ \frac{2}{35}\left(2\eta_2^3 - 15\eta_2^2 - 27\eta_2 + 5\right)f_2^3$$

$$\left. + \frac{2}{7}\left(3\eta_2\eta_4 + 3\eta_2 + 3\eta_4 - 7\right)f_2 f_4\right\}$$

$$+ \frac{1}{6}\left(\frac{3\omega^2}{\pi G\bar{\rho}}\right)^2 (1 - D)\left\{(\eta_2 + 1)f_2 + \frac{3}{7}\eta_2(\eta_2 + 2)f_2^2\right\}$$

$$+ \frac{1}{36}\left(\frac{3\omega^2}{\pi G\bar{\rho}}\right)^3 (1 - D)\left\{\left(\eta_2 + 1\right)f_2\right\}, \tag{2.27}$$

$$a^2 f_4'' + 6D\left(af_4' + f_4\right) - 20f_4$$

$$= \frac{18}{35}\left\{2\eta_2\left(\eta_2 + 2\right) - 3D\left(3\eta_2^2 + 6\eta_2 + 7\right)\right\} f_2^2 \tag{2.28}$$

Continued

$$+\frac{36}{385}\left\{2\left(5\eta_2^2-16\eta_2+3\right)-3D\left(3\eta_2^3+9\eta_2^2+12\eta_2+4\right)\right\}f_2^3$$

$$+\frac{40}{77}\left\{\left(2\eta_2\eta_4+23\eta_2+9\eta_4\right)-9D\left(\eta_2\eta_4+\eta_2+\eta_4\right)\right\}f_2f_4$$

$$+\frac{18}{25025}\left\{2\left(-2917\eta_2^2+10068\eta_2+1020\right)\right.$$

$$\left.-3D\left(429\eta_2^4+858\eta_2^3+2207\eta_2^2+1144\eta_2-2656\right)\right\}f_2^4$$

$$-\frac{2}{5005}\left\{\left(2002\eta_2^2\eta_4+766\eta_2\eta_4-16739\eta_2^2\right.\right.$$

$$\left.+24834\eta_4+111940\eta_2+76386\right)$$

$$\left.+3D\left(2183\eta_2^2\eta_4-1640\eta_2\eta_4+2183\eta_2^2-9560\eta_2-2995\eta_4-181\right)\right\}f_2^2f_4$$

$$+\frac{90}{143}\left\{2\left(\eta_2\eta_6+28\eta_2+4\eta_6\right)-3D\left(3\eta_2\eta_6+3\eta_6+3\eta_2-11\right)\right\}f_2f_6$$

$$+\frac{162}{1001}\left\{2\eta_4\left(\eta_4+30\right)-3D\left(3\eta_4^2+6\eta_4-7\right)\right\}f_4^2$$

$$+\frac{3\omega^2}{\pi G\bar{\rho}}(1-D)\left\{\left(\eta_4+1\right)f_4+\frac{9}{35}\left(3\eta_2^2+6\eta_2+7\right)f_2^2\right.$$

$$\left.+\frac{18}{385}\left(3\eta_2^3+9\eta_2^2+12\eta_2+4\right)f_2^3+\frac{60}{77}\left(\eta_2\eta_4+\eta_2+\eta_4\right)f_2f_4\right\}$$

$$+\frac{1}{6}\left(\frac{3\omega^2}{\pi G\bar{\rho}}\right)^2(1-D)\left\{(\eta_4+1)f_4+\frac{9}{35}\left(3\eta_2^2+6\eta_2+7\right)f_2^2\right\} \tag{2.28}$$

$$a^2f_6''+6D\left(af_6'+f_6\right)-42f_6$$

$$=\frac{18}{77}\left\{4\left(3-\eta_2\right)\left(\eta_2+2\right)-3D\left(\eta_2^3+3\eta_2^2+15\eta_2+5\right)\right\}f_2^3$$

$$+\frac{10}{11}\left\{2\left(\eta_2\eta_4+6\eta_2-\eta_4\right)-3D\left(3\eta_2\eta_4+3\eta_2+3\eta_4+11\right)\right\}f_2f_4$$

$$+\frac{36}{385}\left\{2\left(83\eta_2^2+225\eta_2+150\right)-3D\left(9\eta_2^2+56\eta_2+104\right)\right\}f_2^4$$

$$+\frac{2}{77}\left\{2\left(142-167\eta_2-31\eta_4+59\eta_2^2+13\eta_2\eta_4\right)\right. \tag{2.29}$$

(2.29) Continued

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$$-9D\left(9\eta_2^2\eta_4 + 18\eta_2\eta_4 + 9\eta_2^2 + 10\eta_4 + 62\eta_2 + 24\right)\Big\} f_2^2 f_4$$

$$+\frac{28}{55}\left\{2\eta_2\eta_6 + 6\eta_6 + 45 - 9D\left(\eta_2\eta_6 + \eta_2 + \eta_6\right)\right\} f_2 f_6$$

$$+\frac{20}{99}\left\{2\eta_4\left(\eta_4 + 19\right) - 3D\left(3\eta_4^2 + 6\eta_4 + 4\right)\right\} f_4^2$$

$$+\frac{3\omega^2}{\pi G\bar{\rho}}(1-D)\Big\{\left(\eta_6 + 1\right) f_6 + \frac{5}{11}\left(3\eta_2\eta_4 + 3\eta_2 + 3\eta_4 + 11\right) f_2 f_4$$

$$+\frac{9}{77}\left(\eta_2^3 + 3\eta_2^2 + 15\eta_2 + 5\right) f_2^3\Big\}, \qquad (2.29)$$

and

$$a^2 f_8'' + 6D\left(a f_8' + f_8\right) - 72 f_8$$

$$= \frac{144}{715}\left\{\left(7\eta_2^2 + 2\eta_2 - 24\right) - 3D\left(6\eta_2^2 + 8\right)\right\} f_2^4$$

$$+\frac{28}{143}\Big\{2\left(88 + 2\eta_2 + 6\eta_4 + \eta_2^2 - 8\eta_2\eta_4\right)$$

$$-3D\left(3\eta_2^2\eta_4 + 6\eta_2\eta_4 + 3\eta_2^2 + 15\eta_4 + 54\eta_2 + 23\right)\Big\} f_2^2 f_4$$

$$+\frac{56}{65}\left\{2\left(15\eta_2 - 3\eta_6 + \eta_2\eta_6\right) - 3D\left(3\eta_2\eta_6 + 3\eta_2 + 3\eta_6 + 15\right)\right\} f_2 f_6$$

$$+\frac{490}{1287}\left\{2\eta_4\left(\eta_4 + 4\right) - 3D\left(3\eta_4^2 + 6\eta_4 + 19\right)\right\} f_4^2, \qquad (2.30)$$

where primes on f_j denote differentiation with respect to a and where we have abbreviated

$$\bar{\rho} \equiv \frac{3}{a^3}\int_0^a \rho a^2\, da, \qquad D \equiv \frac{\rho}{\bar{\rho}}. \qquad (2.31)$$

The boundary conditions necessary for the construction of the particular solutions of Eqs. (2.27)-(2.30), which are to represent the amplitudes $f_j(a)$ of the individual harmonics on the right-hand side of the expansion Eq. (2.1) for r', are imposed partly at the center and partly at the boundary of our configuration. As, at the center, all the f_j's are to be a minimum; the necessary conditions for this to be true imply that

$$f_j'(0) = 0, \qquad j = 2, 4, 6, 8, \ldots. \qquad (2.32)$$

On the other hand, at the boundary $a = a_1$ all the E_j's, as given by Eqs. (2.2)-(2.6), vanish; for $j > 0$ the F_j's continue to be given by Eqs. (2.8)-(2.11) or Eqs. (2.22)-(2.25), whereas for $j = 0$, $4\pi F_0(a_1)$ represents by Eq. (2.7) the total mass m_1 of our configuration. Therefore, inserting Eqs. (2.22)-(2.25) in Eqs. (2.18)-(2.21), we find that for $a = a_1$

$$
\begin{aligned}
&2f_2 + a_1 f_2' + \frac{5}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right) \\
&\quad = \frac{2}{7}\left(\eta_2^2 + 3\eta_2 + 6\right)f_2^2 - \frac{4}{35}\left(2\eta_2^3 + 15\eta_2^2 + 30\eta_2 + 38\right)f_2^3 \\
&\qquad + \frac{2}{7}\left(2\eta_2\eta_4 + 3\eta_2 + 3\eta_4 + 26\right)f_2 f_4 \\
&\qquad + \frac{4}{1155}\left(-42\eta_2^4 + 342\eta_2^3 + 609\eta_2^2 + 1175\eta_2 + 1258\right)f_2^4 \\
&\qquad - \frac{4}{77}\left(9\eta_2^2\eta_4 + 24\eta_2\eta_4 + 17\eta_4 + 12\eta_2^2 + 76\eta_2 + 192\right)f_2^2 f_4 \\
&\qquad + \frac{100}{693}\left(\eta_4^2 + 3\eta_4 + 20\right)f_4^2 \\
&\qquad + \frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)\Bigg\{\left(\eta_2 + 5\right)f_2 - \frac{1}{7}\left(2\eta_2^2 + 6\eta_2 + 15\right)f_2^2 \\
&\qquad + \frac{4}{35}\left(2\eta_2^3 + 15\eta_2^2 + 30\eta_2 + 47\right)f_2^3 \\
&\qquad - \frac{2}{7}\left(2\eta_2\eta_4 + 3\eta_2 + 3\eta_4 + 29\right)f_2 f_4\Bigg\}, \qquad (2.33)
\end{aligned}
$$

$$
\begin{aligned}
&4f_4 + a_1 f_4' \\
&\quad = \frac{18}{35}\left(\eta_2 + 2\right)\left(\eta_2 + 3\right)f_2^2 \\
&\qquad - \frac{36}{385}\left(3\eta_2^3 + 18\eta_2^2 + 44\eta_2 + 54\right)f_2^3 \\
&\qquad + \frac{20}{77}\left(2\eta_2\eta_4 + 5\eta_2 + 5\eta_4 + 26\right)f_2 f_4 \\
&\qquad + \frac{18}{25025}\left(367\eta_2^4 + 3427\eta_2^3 + 12167\eta_2^2 + 21185\eta_2 + 22278\right)f_2^4 \\
&\qquad - \frac{1}{5005}\left(4366\eta_2^2\eta_4 + 23470\eta_2\eta_4 + 32289\eta_4\right. \qquad (2.34)
\end{aligned}
$$

Continued

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$$+ 6730\eta_2^2 + 60952\eta_2 + 171744\Big)f_2^2 f_4$$

$$+\frac{45}{143}\Big(2\eta_2\eta_6 + 5\eta_2 + 5\eta_6 + 48\Big)f_2 f_6$$

$$+\frac{162}{1001}\Big(\eta_4^2 + 5\eta_4 + 20\Big)f_4^2$$

$$+\frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)\Bigg\{\Big(\eta_4 + 7\Big)f_4 - \frac{9}{35}\Big(2\eta_2^2 + 10\eta_2 + 21\Big)f_2^2$$

$$+\frac{36}{385}\Big(3\eta_2^3 + 18\eta_2^2 + 44\eta_2 + 72\Big)f_2^3$$

$$-\frac{20}{77}\Big(2\eta_2\eta_4 + 5\eta_2 + 5\eta_4 + 35\Big)f_2 f_4\Bigg\}, \tag{2.34}$$

$$6f_6 + a_1 f_6'$$

$$= -\frac{18}{77}\Big(\eta_2^3 + 8\eta_2^2 + 24\eta_2 + 24\Big)f_2^3$$

$$+\frac{5}{11}\Big(2\eta_2\eta_4 + 7\eta_2 + 7\eta_4 + 26\Big)f_2 f_4$$

$$+\frac{36}{385}\Big(2\eta_2^4 + 18\eta_2^3 + 69\eta_2^2 + 197\eta_2 + 144\Big)f_2^4$$

$$-\frac{2}{77}\Big(27\eta_2^2\eta_4 + 144\eta_2\eta_4 + 72\eta_2^2$$

$$+460\eta_2 + 251\eta_4 + 1152\Big)f_2^2 f_4$$

$$+\frac{14}{55}\Big(2\eta_2\eta_6 + 7\eta_2 + 7\eta_6 + 48\Big)f_2 f_6$$

$$+\frac{20}{99}\Big(\eta_4^2 + 7\eta_4 + 20\Big)f_4^2 + \frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)\Bigg\{\Big(\eta_6 + 9\Big)f_6$$

$$+\frac{18}{77}\Big(\eta_2^3 + 8\eta_2^2 + 24\eta_2 + 39\Big)f_2^3$$

$$-\frac{5}{11}\Big(2\eta_2\eta_4 + 7\eta_2 + 7\eta_4 + 41\Big)f_2 f_4\Bigg\}, \tag{2.35}$$

and

$$
\begin{aligned}
8f_8 &+ a_1 f_8' \\
&= \frac{72}{715}\left(\eta_2^4 + 11\eta_2^3 + 61\eta_2^2 + 121\eta_2 + 110\right)f_2^4 \\
&- \frac{28}{143}\left(3\eta_2^2\eta_4 + 20\eta_2\eta_4 + 10\eta_2^2 + 74\eta_2 + 41\eta_4 + 160\right)f_2^2 f_4 \\
&+ \frac{28}{65}\left(2\eta_2\eta_6 + 9\eta_2 + 9\eta_6 + 48\right)f_2 f_6 \\
&+ \frac{490}{1287}\left(\eta_4 + 4\right)\left(\eta_4 + 5\right)f_4^2 .
\end{aligned}
\tag{2.36}
$$

SOLUTION OF THE EQUATIONS

A construction of the requisite particular solutions of the simultaneous Eqs. (2.27)–(2.30) subject to the boundary conditions Eqs. (2.32)–(2.36) can be accomplished, as in Paper 1, only by successive approximations in the following manner. Within the scheme of a first-order approximation, Eq. (2.27) reduces to the homogeneous equation

$$a^2 f_2'' + 6D(af_2' + f_2) - 6f_2 = 0 \tag{3.1}$$

for f_2, where D continues to be given by Eq. (2.31). The function $D(a)$ depends only on the equilibrium structure of the undistorted configuration and will generally be known to us only in numerical form. By definition, however, $D(0) = 1$, and in the proximity of the origin can be expanded in a series of the form

$$D(a) = 1 - \lambda a^2 + \ldots , \tag{3.2}$$

even powers only occurring on the right-hand side on account of spherical symmetry of our configuration in its undistorted (nonrotating) state.

If so, however, Eq. (3.1) clearly admits, in the proximity of the origin, of a solution varying as

$$f_2 = k_2\left(1 + \frac{3}{7}\lambda a^2 + \ldots\right), \tag{3.3}$$

containing likewise only even powers of a on its right-hand side k_2 being an arbitrary positive constant. Integration of Eq. (3.1) can then proceed (numerically or otherwise) until a boundary is reached at which $a = a_1$ such that $D(a_1) = 0$ and where, by Eq. (2.33), the surface values of f_2 and af_2' should fulfill the algebraic equation

$$2f_2 + a_1 f_2' + \frac{5}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right) = 0 \tag{3.4}$$

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to quantities of first order in superficial distortion. On account of the homogeneous nature of Eq. (2.1), the values of f_2, as well as resulting from our integration, are proportional to k_2; if so, the algebraic Eq. (3.4) valid for $a = a_1$ can then be used to specify the value of the nondimensional parameter $\omega^2 a_1^3/Gm_1$ corresponding to the initially adopted value of k_2. Conversely, should we wish to obtain the values of $f_j(a)$ corresponding to a specific value of $\omega^2 a_1^3/Gm_1$, our solution, started with an arbitrary value of k_2 (say, $k_2 = 1$), should be accordingly scaled down at this stage.

With a first-order approximation to $f_2(a)$ thus in our hands, we can now proceed to the second-order approximation, which consists of finding a solution of the equations

$$
\begin{aligned}
a^2 f_2'' &+ 6D(af_2' + f_2) - 6f_2 \\
&= \frac{2}{7}\left\{2\eta_2\left(\eta_2 + 9\right) - 9\eta_2\left(\eta_2 + 2\right)\right\} f_2^2 \\
&+ \frac{3\omega^2}{\pi G\bar{\rho}}(1 - D)\left(\eta_2 + 1\right) f_2
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
a^2 f_4'' &+ 6D\left(af_4' + f_4\right) - 20f_4 \\
&= \frac{18}{35}\left\{2\eta_2\left(\eta_2 + 2\right) - 3D\left(3\eta_2^2 + 6\eta_2 + 7\right)\right\} f_2^2 ,
\end{aligned}
\tag{3.6}
$$

subject to the boundary conditions requiring that at the center

$$f_2'(0) = f_4'(0) = 0 , \tag{3.7}$$

while on the surface $a = a_1$,

$$2f_2 + a_1 f_2' + \frac{5}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right) = \frac{1}{7}\left\{2\eta_2^2 + 6\eta_2 + 12\right\} f_2^2 + \frac{2}{3}\left(\frac{\omega^2 a_1^3}{Gm_1}\right)\left(\eta_2 + 5\right) f_2^2 , \tag{3.8}$$

and

$$4f_4 + a_1 f_4' = \frac{18}{35}\left(\eta_2^2 + 5\eta_2 + 6\right) f_2^2 . \tag{3.9}$$

As Eq. (3.5) is independent of f_4, its solution satisfying Eq. (3.7) can be expanded near the origin in a series in ascending even powers of a. If, moreover, we note that

$$\frac{3\omega^2}{\pi G\bar{\rho}} = \frac{D}{\dfrac{\rho}{\rho_c}}\left(\frac{3\omega^2}{\pi G\rho_c}\right), \tag{3.10}$$

where $\rho_c \equiv \rho(0)$ and, consistent with Eq. (3.2),

$$\frac{\rho}{\rho_c} = 1 - \frac{5}{2}\lambda a^2 + \ldots , \tag{3.11}$$

we find that, correctly to terms of the order of the squares of superficial distortion,

$$f_2 = k_2\left\{1 + \frac{3}{7}(1+v)\lambda a^2 + \frac{3}{14}\left(1 - \frac{11}{5}v\right)\lambda^2 a^4 - \frac{4k_2}{147}\lambda^2 a^4 + \ldots\right\}, \tag{3.12}$$

where

$$v = \frac{\omega^2}{2\pi G\rho_c} \tag{3.13}$$

denotes a constant.*

Moreover, the structure of Eq. (3.6) for f_4, solved in a similar way [2], discloses that near the origin

$$\begin{aligned} f_4 = {} & k_4 a^2\left\{1 + \frac{18}{55}\lambda a^2 + \ldots\right\} \\ & + \frac{27}{35}k_2^2\left\{1 - \frac{82}{77}\lambda^2 a^4 + \frac{8}{99}k_2\lambda^2 a^4 \right. \\ & \left. + \frac{156}{385}k_2^{-1}(1+v)\lambda^2 a^4 + \ldots\right\}, \end{aligned} \tag{3.14}$$

consisting of two parts: the first (factored by k_4) represents the "complementary function" of the homogeneous version of Eq. (3.6) with its left-hand side equated to zero, and the second (factored by k_2^2) stands for a "particular integral" arising from a non-vanishing right-hand side. The constant k_4 introduced through the complementary function is new, and its value must be specified (after integration has been completed) from the boundary condition Eq. (3.9), just as k_2 needs to be recomputed from Eq. (3.8).

The foregoing example makes it clear how to extend this procedure to solve for each amplitude $f_j(a)$ of jth harmonic distortion to the requisite degree of accuracy. The structure of the differential equations of the form of Eqs. (2.27)-(2.30) governing these makes it evident that near the origin ($a = 0$) the complementary function of each f_j will vary as $k_j a^{j-2}$, and its particular integral will be factored by $k_2^{j/2}$. In more specific terms, if we set, by definition,

$$f_2(0) = k_2 , \tag{3.15}$$

it follows from the structure of Eqs. (2.27)-(2.30) in which $\eta_j(0) = 0$, that

*Strictly speaking, one should still augment the right-hand side of Eq. (3.12) by quantities arising from possible biquadratic terms on the right-hand side of Eq. (3.2) or (3.11) which are not spelled out explicitly in the latter, their inclusion is left as an exercise for the interested reader.

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$$f_4(0) = \frac{27}{35} f_2^2(0) + \frac{108}{2695} f_2^3(0) - \frac{90072}{175175} f_2^4(0) + \frac{75843}{35035} f_2^2(0) f_4(0)$$

$$- \frac{135}{91} f_2(0) f_6(0) - \frac{243}{1001} f_4^2(0)$$

$$= \frac{27}{35} \left\{ k_2^2 + \frac{4}{77} k_2^3 + \frac{1311}{7007} k_2^4 + \ldots \right\}, \tag{3.16}$$

$$f_6(0) = \frac{5}{6} f_2(0) f_4(0) - \frac{9}{154} f_2^3(0) + \frac{129}{385} f_2^4(0) + \frac{20}{297} f_4^2(0)$$

$$- \frac{34}{693} f_2^2(0) f_4(0)$$

$$= \frac{45}{77} \left\{ k_2^3 + \frac{111}{175} k_2^4 + \ldots \right\}, \tag{3.17}$$

and

$$f_8(0) = \frac{1152}{7865} f_2^4(0) - \frac{1498}{4719} f_2^2(0) f_4(0) + \frac{84}{143} f_2(0) f_6(0) + \frac{9310}{14157} f_4^2(0)$$

$$= \frac{63}{143} k_2^4 + \ldots, \tag{3.18}$$

correctly to quantities of fourth order; the lowest nonzero derivatives of the $f_j(a)$'s which do not vanish at the origin are

$$f_2''(0) = \frac{6}{7} k_2 (1 + v) \lambda, \tag{3.19}$$

and for $j > 2$,

$$f_j^{(j-2)}(0) = (j-2)! k_j, \quad j = 4, 6, 8, \ldots. \tag{3.20}$$

The constants $k_j (j = 2, 4, 6, 8, \ldots)$ constitute the "eigenparameters" of our problem, and their values must be determined with the aid of the boundary conditions Eqs. (2.33)-(2.36) valid at $a = a_1$. Inasmuch as the right-hand sides of Eqs. (2.27)-(2.30) governing f_j are known algebraic functions of $f_2, f_4, \ldots f_{j-2}$, the solution of the entire system evidently lends itself to a solution by successive approximations: first solving for f_2 to accuracy of first order; next solving for f_2 and f_4 to quantities of second order; then continuing until a solution for the f's accurate to jth order of accuracy has been established.

REFERENCES

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